Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka, Surathkal

sam.nitk.ac.in nitksam@gmail.com

Advanced Linear Algebra (MA 409)
Problem Sheet - 7

The Matrix Representation of a Linear Transformation

---

1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $T, U : V \to W$ are linear transformations.

   (a) $[T]_\beta^\gamma = [U]_\beta^\gamma$ implies that $T = U$.

   (b) If $m = \dim(V)$ and $n = \dim(W)$, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

   (c) $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

   (d) $\mathcal{L}(V, W)$ is a vector space.

   (e) $\mathcal{L}(V, W)$=$\mathcal{L}(W, V)$.

2. Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.

   (c) $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$.

   (d) $T : \mathbb{R}^n \to \mathbb{R}$ defined by $T(a_1, a_2, \ldots, a_n) = a_1 + a_n$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1, 2), (2, 3)\}$, compute $[T]_\alpha^\gamma$.

4. Define

$$T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R}) \text{ by } T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b) + (2d)x + bx^2.$$

   Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \text{ and } \gamma = \{1, x, x^2\}.$$

   Compute $[T]_\beta^\gamma$.

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\},$$

and

$$\gamma = \{1\}.$$

(a) Define $T : M_{2\times 2}(F) \to M_{2\times 2}(F)$ by $T(A) = A^t$. Compute $[T]_\alpha$.

(b) Define

$$T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \text{ by } T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[T]_\beta^\alpha$.

(c) Define $T : M_{2\times 2}(F) \to F$ by $T(A) = tr(A)$. Compute $[T]_\alpha^\gamma$.

(d) Define $T : P_2(\mathbb{R}) \to \mathbb{R}$ by $T(f(x)) = f(2)$. Compute $[T]_\beta^\gamma$.

(e) If

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix},$$

compute $[A]_\alpha$.

(f) If $f(x) = 3 - 6x + x^2$, compute $[f(x)]_\beta$.

(g) For $a \in F$, compute $[a]_\gamma$.

6. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

7. Let V be the vector space of complex numbers over the field $\mathbb{R}$. Define $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[T]_\beta$, where $\beta = \{1, i\}$. (Recall that T is not linear if V is regarded as a vector space over the field $\mathbb{C}$.)

8. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = 0$. Then there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[T]_\beta$.

9. Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear transformation. Suppose that W is a T-invariant subspace of V having dimension k. Show that there is a basis β for V such that $[T]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.

10. Let V be a finite-dimensional vector space and T be the projection on W along W', where W and W' are subspaces of V. Find an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

11. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

12. Let $V = P(\mathbb{R})$, and for $j \geq 1$ define $T_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivative of $f(x)$. Prove that the set $\{T_1, T_2, \ldots, T_n\}$ is a linearly independent subset of $\mathcal{L}(V)$ for any positive integer n.

13. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{T \in \mathcal{L}(V, W) : T(x) = 0 \text{ for all x} \in S\}$. Prove the following statements.

    (a) S^0 is a subspace of $\mathcal{L}(V, W)$.

    (b) If S_1 and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.

    (c) If V_1 and V_2 are subspaces of V, then $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.

14. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

*****